

# Irreducible Generic Gelfand–Tsetlin Modules of $\mathfrak{gl}(n)^*$

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**Abstract.** We provide a classification and explicit bases of tableaux of all irreducible generic Gelfand–Tsetlin modules for the Lie algebra  $\mathfrak{gl}(n)$ .

*Key words:* Gelfand–Tsetlin modules; Gelfand–Tsetlin basis; tableaux realization

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## 1 Introduction

Let  $\mathfrak{g}$  be a complex finite-dimensional semisimple Lie algebra. The category of weight modules of  $\mathfrak{g}$  is interesting on its own on the one hand, and it contains some fundamental subcategories like the category  $\mathcal{O}$ , categories of parabolically induced modules, Harish-Chandra modules on the other. A weight  $\mathfrak{g}$ -module is a module which is a direct sum of simple  $\mathfrak{h}$ -modules, where  $\mathfrak{h}$  is a fixed Cartan subalgebra of  $\mathfrak{g}$ . The classification of the simple weight modules is a very hard problem which is solved only for  $\mathfrak{g} = \mathfrak{sl}(2)$ . However, the classification of the simple objects is known for various subcategories of weight modules, including those with finite weight multiplicities [5, 17].

The classification of the simple weight  $\mathfrak{sl}(2)$ -modules involves two parameters that correspond to eigenvalues of the generators of a maximal commutative subalgebra of  $U(\mathfrak{sl}(2))$ , the *Gelfand–Tsetlin subalgebra*. Such subalgebra can be defined for any  $\mathfrak{sl}(n)$  and has a joint spectrum on every finite-dimensional module. This observation leads naturally to the definition of a *Gelfand–Tsetlin module*: a module that is the direct sum of its common generalized eigenspaces with respect to the Gelfand–Tsetlin subalgebra  $\Gamma$ . Such modules were introduced in [2, 3, 4]. Note that an irreducible Gelfand–Tsetlin module does not need to be  $\Gamma$ -diagonalizable [6].

Gelfand–Tsetlin subalgebras and modules appear in various contexts. Such subalgebras were considered in [22] in connection with subalgebras of maximal Gelfand–Kirillov dimension in the universal enveloping algebra of a simple Lie algebra. Furthermore, Gelfand–Tsetlin subalgebras are related to: general hypergeometric functions on the complex Lie group  $GL(n)$  [13, 14]; solutions of the Euler equation [22]; and problems in classical mechanics in general [15, 16].

One natural question is to attempt the classification of all irreducible Gelfand–Tsetlin modules of  $\mathfrak{sl}(n)$ . An explicit construction of all irreducible Gelfand–Tsetlin modules for the case  $n = 3$  was recently obtained in [10]. Various partial results for  $\mathfrak{sl}(3)$  were previously obtained in [1, 6, 7, 8, 9].

A *generic Gelfand–Tsetlin module* is a module spanned by tableaux with noninteger differences of entries in each row (see Definition 5.1). The present paper provides a classification of all irreducible generic Gelfand–Tsetlin modules of  $\mathfrak{sl}(n)$  extending the result in [21] for  $n = 3$ .

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For simplicity we work with  $\mathfrak{gl}(n)$  instead of  $\mathfrak{sl}(n)$ . We also obtain an explicit construction of all irreducible generic modules providing a Gelfand–Tsetlin type basis.

The organization of the paper is as follows. In Section 3 we introduce some basic definitions and preparatory results on Gelfand–Tsetlin modules. In Section 4 we list the Gelfand–Tsetlin formulas and use them to recall the classical result of Gelfand and Tsetlin for finite-dimensional  $\mathfrak{gl}(n)$ -modules. In Section 5 we introduce the notion of generic Gelfand–Tsetlin module and recall the classification of irreducible generic Gelfand–Tsetlin modules of  $\mathfrak{gl}(3)$ . The main theorem in the paper, the classification of irreducible generic Gelfand–Tsetlin  $\mathfrak{gl}(n)$ -modules, is included in Section 6. In the last section we compute the number of irreducible Gelfand–Tsetlin modules in the so-called generic blocks.

## 2 Notation and conventions

Throughout the paper we fix an integer  $n \geq 2$ . The ground field will be  $\mathbb{C}$ . For  $a \in \mathbb{Z}$ , we write  $\mathbb{Z}_{\geq a}$  for the set of all integers  $m$  such that  $m \geq a$ . Similarly, we define  $\mathbb{Z}_{< a}$ , etc. By  $\mathfrak{gl}(n)$  we denote the general linear Lie algebra consisting of all  $n \times n$  complex matrices, and by  $\{E_{i,j} \mid 1 \leq i, j \leq n\}$ , the standard basis of  $\mathfrak{gl}(n)$  of elementary matrices. We fix the standard Cartan subalgebra  $\mathfrak{h}$ , the standard triangular decomposition and the corresponding basis of simple roots of  $\mathfrak{gl}(n)$ . The weights of  $\mathfrak{gl}(n)$  will be written as  $n$ -tuples  $(\lambda_1, \dots, \lambda_n)$ .

For a Lie algebra  $\mathfrak{a}$  by  $U(\mathfrak{a})$  we denote the universal enveloping algebra of  $\mathfrak{a}$ . Throughout the paper  $U = U(\mathfrak{gl}(n))$ . For a commutative ring  $R$ , by  $\text{Specm } R$  we denote the set of maximal ideals of  $R$ .

We will write the vectors in  $\mathbb{C}^{\frac{n(n+1)}{2}}$  in the following form:

$$L = (l_{ij}) = (l_{n1}, \dots, l_{nn} \mid \dots \mid l_{21}, l_{22} \mid l_{11}).$$

For  $1 \leq j \leq i \leq n$ ,  $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$  is defined by  $(\delta^{ij})_{ij} = 1$  and all other  $(\delta^{ij})_{kl}$  are zero.

For  $i > 0$  by  $S_i$  we denote the  $i$ th symmetric group. Throughout the paper we set  $G := S_n \times \dots \times S_1$ .

## 3 Gelfand–Tsetlin modules

Recall that  $U = U(\mathfrak{gl}(n))$ . Let for  $m \leq n$ ,  $\mathfrak{gl}_m$  be the Lie subalgebra of  $\mathfrak{gl}(n)$  spanned by  $\{E_{ij} \mid i, j = 1, \dots, m\}$ . We have the following chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_n.$$

It induces the chain  $U_1 \subset U_2 \subset \dots \subset U_n$  for the universal enveloping algebras  $U_m = U(\mathfrak{gl}_m)$ ,  $1 \leq m \leq n$ . Let  $Z_m$  be the center of  $U_m$ . The subalgebra of  $U$  generated by  $\{Z_m \mid m = 1, \dots, n\}$  will be called the (*standard*) *Gelfand–Tsetlin subalgebra* of  $U$  and will be denoted by  $\Gamma$  [2].

**Definition 3.1.** A finitely generated  $U$ -module  $M$  is called a *Gelfand–Tsetlin module* (with respect to  $\Gamma$ ) if

$$M = \bigoplus_{\mathfrak{m} \in \text{Specm } \Gamma} M(\mathfrak{m}),$$

where  $M(\mathfrak{m}) = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \geq 0\}$ .

For each  $\mathfrak{m} \in \text{Specm } \Gamma$  we have associated a character  $\chi_{\mathfrak{m}} : \Gamma \rightarrow \Gamma/\mathfrak{m} \sim \mathbb{C}$ . In the same way, for each non-zero character  $\chi : \Gamma \rightarrow \mathbb{C}$  we have that  $\text{Ker}(\chi)$  is a maximal ideal of  $\Gamma$ . So, we have

a natural identification between characters of  $\Gamma$  and elements of  $\text{Specm } \Gamma$ . Using characters we can define Gelfand–Tsetlin modules. A  $U$ -module  $M$  is called *Gelfand–Tsetlin module* (with respect to  $\Gamma$ ) if

$$M = \bigoplus_{\chi \in \Gamma^*} M(\chi),$$

where  $M(\chi) = \{v \in M : \forall g \in \Gamma, \exists k \in \mathbb{Z}_{>0} \text{ such that } (g - \chi(g))^k v = 0\}$ . The *Gelfand–Tsetlin support* of  $M$  is the set  $\text{Supp}_{\text{GT}}(M) := \{\chi \in \Gamma^* : M(\chi) \neq 0\}$ .

**Lemma 3.2.** *Any submodule of a Gelfand–Tsetlin module over  $\mathfrak{gl}(n)$  is a Gelfand–Tsetlin module.*

**Proof.** The proof is standard, but for a sake of completeness, we provide the important details. Let  $M$  be a Gelfand–Tsetlin  $\mathfrak{gl}(n)$ -module and  $N$  any submodule of  $M$ . We will prove that, if  $\{\chi_1, \dots, \chi_k\}$  is a set of distinct Gelfand–Tsetlin characters in  $\text{Supp}_{\text{GT}}(M)$  such that  $\sum_{i=1}^k v_i \in N$  with  $v_i \in M(\chi_i)$ , then  $v_i \in N$  for all  $i = 1, \dots, k$ .

Without loss of generality we assume that  $k = 2$ . Since  $\chi_1 \neq \chi_2$ , there exist  $g \in \Gamma$  and  $r \leq s$  in  $\mathbb{Z}_{\geq 0}$  such that  $\chi_1(g) \neq \chi_2(g)$ ,  $(g - \chi_1(g))^r(v_1) = 0$  and  $(g - \chi_2(g))^s(v_2) = 0$ . Let  $a := \chi_1(g)$  and  $b := \chi_2(g)$ . Then, if  $w = v_1 + v_2$  we have  $(g - b)^s w = (g - b)^s v_1 \in N$ . Let  $y := (g - b)^s v_1$ . We have that  $y \in N$  on one hand and

$$y = ((g - a) + (a - b))^s v_1 = \sum_{k=0}^{s-1} \binom{s}{k} (a - b)^{s-k} (g - a)^k v_1 \in N$$

on the other. As  $\binom{s}{k} (a - b)^{s-k} \neq 0$  for any  $k$ , using that  $(g - a)^{r-1} y \in N$ , we obtain  $(g - a)^{r-1} v_1 \in N$ . Reasoning in the same way, from  $(g - a)^{r-i} y \in N$ , and  $(g - a)^{r-1} v_1, \dots, (g - a)^{r-i+1} v_1 \in N$  we obtain  $x^{r-i} v_1 \in N$ . Hence  $v_1 \in N$  and consequently,  $v_2 \in N$ . ■

One can choose the following generators of  $\Gamma$ :  $\{c_{mk} \mid 1 \leq k \leq m \leq n\}$ , where

$$c_{mk} = \sum_{(i_1, \dots, i_k) \in \{1, \dots, m\}^k} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1}. \quad (3.1)$$

Let  $\Lambda$  be the polynomial algebra in the variables  $\{\lambda_{ij} \mid 1 \leq j \leq i \leq n\}$ . The action of the symmetric group  $S_i$  on  $\{\lambda_{ij} \mid 1 \leq j \leq i\}$  induces the action of  $G = S_n \times \cdots \times S_1$  on  $\Lambda$ . There is a natural embedding  $\iota : \Gamma \longrightarrow \Lambda$  given by  $\iota(c_{mk}) = \gamma_{mk}(\lambda)$ , where

$$\gamma_{mk}(\lambda) = \sum_{i=1}^m (\lambda_{mi} + m - 1)^k \prod_{j \neq i} \left(1 - \frac{1}{\lambda_{mi} - \lambda_{mj}}\right). \quad (3.2)$$

Hence,  $\Gamma$  can be identified with  $G$ -invariant polynomials in  $\Lambda$ .

**Remark 3.3.** In what follows, we will identify the set  $\text{Specm } \Lambda$  of maximal ideals of  $\Lambda$  with the set  $\mathbb{C}^{\frac{n(n+1)}{2}}$ . Then we have a surjective map  $\pi : \text{Specm } \Lambda \rightarrow \text{Specm } \Gamma$ . Moreover, since  $\Lambda$  is integral over  $\Gamma$ , there are finitely many maximal ideals of  $\Lambda$  that map to a fixed maximal ideal of  $\Gamma$ . The different maximal ideals of  $\Lambda$  are obtained from each other under permutations in the group  $G$ .

If  $\pi(\ell) = \mathfrak{m}$  for some  $\ell \in \text{Specm } \Lambda$ , then we write  $\ell = \ell_{\mathfrak{m}}$  and say that  $\ell_{\mathfrak{m}}$  is *lying over*  $\mathfrak{m}$ .

## 4 Finite-dimensional modules of $\mathfrak{gl}(n)$

In this section we recall a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite-dimensional  $\mathfrak{gl}(n)$ -module.

**Definition 4.1.** For a vector  $L = (l_{ij})$  in  $\mathbb{C}^{\frac{n(n+1)}{2}}$ , by  $T(L)$  we will denote the following array with entries  $\{l_{ij} : 1 \leq j \leq i \leq n\}$

$$\begin{array}{ccccccc}
 \boxed{l_{n1}} & \boxed{l_{n2}} & & \cdots & & \boxed{l_{n,n-1}} & \boxed{l_{nn}} \\
 & \boxed{l_{n-1,1}} & & \cdots & & \boxed{l_{n-1,n-1}} & \\
 & & \cdots & & \cdots & & \cdots \\
 & & & \boxed{l_{21}} & \boxed{l_{22}} & & \\
 & & & & \boxed{l_{11}} & & 
 \end{array}$$

Such an array will be called a *Gelfand–Tsetlin tableau* of height  $n$ . A Gelfand–Tsetlin tableau of height  $n$  is called *standard* if  $l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0}$  and  $l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}$  for all  $1 \leq i \leq k \leq n-1$ .

Note that, for sake of convenience, the second condition above is slightly different from the original condition in [12].

**Theorem 4.2** ([12]). *Let  $L(\lambda)$  be the finite-dimensional irreducible module over  $\mathfrak{gl}(n)$  of highest weight  $\lambda = (\lambda_1, \dots, \lambda_n)$ . Then there exist a basis of  $L(\lambda)$  consisting of all standard tableaux  $T(L) = T(l_{ij})$  with fixed top row  $l_{nj} = \lambda_j - j + 1$ . Moreover, the action of the generators of  $\mathfrak{gl}(n)$  on  $L(\lambda)$  is given by the Gelfand–Tsetlin formulas:*

$$\begin{aligned}
 E_{k,k+1}(T(L)) &= - \sum_{i=1}^k \left( \frac{\prod_{j=1}^{k+1} (l_{ki} - l_{k+1,j})}{\prod_{j \neq i}^k (l_{ki} - l_{kj})} \right) T(L + \delta^{ki}), \\
 E_{k+1,k}(T(L)) &= \sum_{i=1}^k \left( \frac{\prod_{j=1}^{k-1} (l_{ki} - l_{k-1,j})}{\prod_{j \neq i}^k (l_{ki} - l_{kj})} \right) T(L - \delta^{ki}), \\
 E_{kk}(T(L)) &= \left( k - 1 + \sum_{i=1}^k l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L),
 \end{aligned} \tag{4.1}$$

if the new tableau  $T(L \pm \delta^{ki})$  is not standard, then the corresponding summand of  $E_{k,k+1}(T(L))$  or  $E_{k+1,k}(T(L))$  is zero by definition. Furthermore, for  $s \leq r$ ,

$$c_{rs}(T(L)) = \gamma_{rs}(l)T(L), \tag{4.2}$$

where  $\{c_{rs}\}$  are the generators of  $\Gamma$  defined in (3.1) and  $\gamma_{rs}$  are defined in (3.2) (see [23]).

The formulas above are called *Gelfand–Tsetlin formulas* for  $\mathfrak{gl}(n)$ . These formulas were extended to the case of  $U_q(\mathfrak{gl}(n))$  in [19].

## 5 Generic Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$

Theorem 4.2 gives an explicit realization of any irreducible finite-dimensional  $\mathfrak{gl}(n)$ -module. Using the Gelfand–Tsetlin formulas, Drozd, Futorny and Ovsienko defined the class of infinite-dimensional generic modules for  $\mathfrak{gl}(n)$  in [2].

**Definition 5.1.** A Gelfand–Tsetlin tableau  $T(L)$  (equivalently,  $L \in \mathbb{C}^{\frac{n(n+1)}{2}}$ ) is called *generic* if  $l_{ki} - l_{kj} \notin \mathbb{Z}$  for all  $1 \leq i \neq j \leq k \leq n-1$ . A character  $\chi$  and  $\mathfrak{n} = \text{Ker } \chi$  are called *generic* if  $\ell_{\mathfrak{n}}$  is generic for one choice (hence for all choices) of  $\ell_{\mathfrak{n}}$  lying over  $\mathfrak{n}$ . A Gelfand–Tsetlin module  $M$  will be called a *generic Gelfand–Tsetlin module* if every  $\mathfrak{n}$  in  $\text{Supp}_{\text{GT}}(M)$  is generic.

**Theorem 5.2** ([2, Section 2.3] and [18, Theorem 2]). *Let  $T(L) = T(l_{ij})$  be a generic Gelfand–Tsetlin tableau of height  $n$ . Denote by  $\mathcal{B}(T(L))$  the set of all Gelfand–Tsetlin tableaux  $T(R) = T(r_{ij})$  satisfying  $r_{nj} = l_{nj}$ ,  $r_{ij} - l_{ij} \in \mathbb{Z}$  for  $1 \leq j \leq i \leq n-1$ .*

- (i) *The vector space  $V(T(L)) = \text{span } \mathcal{B}(T(L))$  has a structure of a  $\mathfrak{gl}(n)$ -module with action of the generators of  $\mathfrak{gl}(n)$  given by the Gelfand–Tsetlin formulas (4.1).*
- (ii) *The action of the generators of  $\Gamma$  on the basis elements of  $V(T(L))$  is given by (4.2).*
- (iii) *The  $\mathfrak{gl}(n)$ -module  $V(T(L))$  is a Gelfand–Tsetlin module all of whose Gelfand–Tsetlin multiplicities are 1.*

**Remark 5.3.** The basis of the module in the previous theorem is

$$\mathcal{B}(T(L)) = \{T(L + z) : z \in \mathbb{Z}^{\frac{n(n+1)}{2}} \text{ and } z_{n1} = \cdots = z_{nn} = 0\}.$$

By a slight abuse of notation we will identify elements in  $\mathbb{Z}^{\frac{n(n-1)}{2}}$  with elements  $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$  such that  $z_{n1} = \cdots = z_{nn} = 0$ . This will allow us to write  $T(L + z)$  for  $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ .

**Remark 5.4.** In what follows, we will apply Lemma 3.2 and use that the elements of  $\Gamma$  separate the tableaux in the submodules of  $V(T(L))$  in the following sense. Let  $N$  be a  $\mathfrak{gl}(n)$ -submodule of  $V(T(L))$ ,  $g \in \mathfrak{gl}(n)$ , and  $T(R)$  be a tableau in  $N$ . Then, if  $g \cdot T(R) = \sum_i c_i T(R_i)$  for some distinct tableaux  $T(R_i)$  in  $\mathcal{B}(T(L))$  and nonzero  $c_i \in \mathbb{C}$ , we have  $T(R_i) \in N$  for all  $i$ .

**Theorem 5.5.** *If  $\mathfrak{n} \in \text{Specm } \Gamma$  is generic, then there exists a unique irreducible Gelfand–Tsetlin module  $N$  such that  $N(\mathfrak{n}) \neq 0$ .*

**Proof.** Let  $X_{\mathfrak{n}} = U/U\mathfrak{n}$ . We know that  $X_{\mathfrak{n}} = U/U\mathfrak{n}$  is a Gelfand–Tsetlin module. Furthermore, any irreducible Gelfand–Tsetlin module  $M$  with  $M(\mathfrak{n}) \neq 0$  is a homomorphic image of  $X_{\mathfrak{n}}$ , and  $X_{\mathfrak{n}}(\mathfrak{n})$  maps onto  $M(\mathfrak{n})$ . Since both spaces  $X_{\mathfrak{n}}(\mathfrak{n})$  and  $M(\mathfrak{n})$  are  $\Gamma$ -modules then the projection  $X_{\mathfrak{n}}(\mathfrak{n}) \rightarrow M(\mathfrak{n})$  is a homomorphism of  $\Gamma$ -modules (see also [11, Corollary 5.3]). Taking into account that  $\dim X_{\mathfrak{n}}(\mathfrak{n}) \leq 1$ , we conclude that  $X_{\mathfrak{n}}$  has a unique maximal submodule (which does not intersect  $X_{\mathfrak{n}}(\mathfrak{n})$ ) and hence there exist a unique irreducible module  $N$  with  $N(\mathfrak{n}) \neq 0$ . ■

**Definition 5.6.** If  $T(R)$  is a generic tableau and  $\mathfrak{r} \in \text{Specm } \Gamma$  corresponds to  $R$  then, the unique module  $N$  such that  $N(\mathfrak{r}) \neq 0$  is called the *irreducible Gelfand–Tsetlin module containing  $T(R)$* , or simply, the *irreducible module containing  $T(R)$* .

Our goal is to describe explicitly the irreducible Gelfand–Tsetlin module containing  $T(R)$  for every generic tableau  $T(R)$ . Below we recall how this is achieved in the case  $n = 3$  in [20]. One should note that the methods used in [20] involve direct computations based on a case-by-case consideration, while in the present paper we provide an invariant proof. Also, we reformulate the result in [20] in terms of  $T(L + z)$ .

For any tableau  $T(R) \in \{T(L+z) : z \in \mathbb{Z}^3\}$  and any  $1 < p \leq 3$ ,  $1 \leq s \leq p$ , and  $1 \leq u \leq p-1$ , define

$$\Omega^+(T(R)) := \{(p, s, u) : r_{p,s} - r_{p-1,u} \in \mathbb{Z}_{\geq 0}\}.$$

**Theorem 5.7** ([20]). *If  $T(L)$  is a generic Gelfand–Tsetlin tableau of height 3, then the following is a basis for the irreducible  $\mathfrak{gl}(3)$ -module containing  $T(L)$ :*

$$\mathcal{I}(T(L)) := \{T(L+z) : z \in \mathbb{Z}^3 \text{ and } \Omega^+(T(L)) = \Omega^+(T(L+z))\}.$$

The action of  $\mathfrak{gl}(3)$  on this irreducible module is given by the Gelfand–Tsetlin formulas.

**Example 5.8.** Consider  $a, b, c \in \mathbb{C}$  such that  $\{a-b, a-c, b-c\} \cap \mathbb{Z} = \emptyset$ ,  $L = (a, b, c|a, b+1|a)$  and

$$T(L) = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline a & b+1 & \\ \hline a & & \\ \hline \end{array}$$

then  $\Omega^+(T(L)) = \{(3, 1, 1), (2, 1, 1)\}$ . So, by Theorem 5.7, the irreducible module containing  $T(L)$  has basis

$$\mathcal{I}(T(L)) = \{T(L + (m, n, k)) : (m, n, k) \in \mathbb{Z}^3, m \leq 0, k \leq m, \text{ and } n > -1\}.$$

## 6 Classification of irreducible generic Gelfand–Tsetlin $\mathfrak{gl}(n)$ -modules

In this section we prove the main result in the paper, i.e. the generalization of Theorem 5.7 for  $\mathfrak{gl}(n)$ . For convenience we introduce and recall some notation.

**Notation 6.1.** Let  $T(L) = T(l_{ij})$  be a fixed tableau of height  $n$ .

- (i)  $\mathcal{B}(T(L)) := \{T(L+z) : z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\}.$
- (ii)  $V(T(L)) := \text{span } \mathcal{B}(T(L)).$
- (iii) For any  $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$  and for any  $1 < p \leq n$ ,  $1 \leq s \leq p$  and  $1 \leq u \leq p-1$  we define:
  - (a)  $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u};$
  - (b)  $\Omega(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \mathbb{Z}\};$
  - (c)  $\Omega^+(T(R)) := \{(p, s, u) : \omega_{p,s,u}(T(R)) \in \mathbb{Z}_{\geq 0}\};$
  - (d)  $\mathcal{N}(T(R)) := \{T(Q) \in \mathcal{B}(T(L)) : \Omega^+(T(R)) \subseteq \Omega^+(T(Q))\};$
  - (e)  $W(T(R)) := \text{span } \mathcal{N}(T(R));$
  - (f)  $U \cdot T(R)$ : the  $\mathfrak{gl}(n)$ -submodule of  $V(T(L))$  generated by  $T(R)$ .

### 6.1 Basis for the module generated by a single tableau

In order to find an explicit basis of every irreducible generic module, we first find a basis of  $U \cdot T(R)$  for any tableau  $T(R)$  in  $\mathcal{B}(T(L))$ .

**Proposition 6.2.** *For any  $T(R) \in \mathcal{B}(T(L))$ , the Gelfand–Tsetlin formulas endow  $W(T(R))$  with a  $\mathfrak{gl}(n)$ -module structure.*

**Proof.** It is enough to prove  $U \cdot T(Q) \subseteq W(T(R))$  for any  $T(Q) = T(q_{ij}) \in \mathcal{N}(T(R))$ . We will show  $g \cdot T(Q)$  is in  $W(T(R))$  for every (standard) generator  $g$  of  $\mathfrak{gl}(n)$ .

Suppose  $g = E_{k,k+1}$  for some  $1 \leq k \leq n-1$ . By the Gelfand–Tsetlin formulas, we have

$$E_{k,k+1}(T(Q)) = - \sum_{i=1}^k \left( \frac{\prod_{j=1}^{k+1} (q_{ki} - q_{k+1,j})}{\prod_{j \neq i}^k (q_{ki} - q_{kj})} \right) T(Q + \delta^{ki}).$$

If  $E_{k,k+1}(T(Q)) \notin W(T(R))$ , then there exist  $k$  and  $i$  such that  $T(Q) \in \mathcal{N}(T(R))$  but  $T(Q + \delta^{ki}) \notin \mathcal{N}(T(R))$ . That implies

$$\Omega^+(T(R)) \subseteq \Omega^+(T(Q)) \text{ and } \Omega^+(T(R)) \not\subseteq \Omega^+(T(Q + \delta^{ki})).$$

Hence, there exists  $(p, s, u) \in \Omega^+(T(R))$  such that  $\omega_{p,s,u}(T(Q)) \in \mathbb{Z}_{\geq 0}$  and  $\omega_{p,s,u}(T(Q + \delta^{ki})) \notin \mathbb{Z}_{\geq 0}$ . The latter holds only in two cases:

$$(p, s, u) \in \{(k, i, u), (k+1, s, i) : 1 \leq u \leq k-1; 1 \leq s \leq k+1\}.$$

Note that if neither of these two cases hold, we have  $\omega_{p,s,u}(T(Q + \delta^{ki})) = \omega_{p,s,u}(T(Q))$ . We consider now each of the two cases separately.

- (i) Suppose  $(p, s, u) = (k, i, u)$ . Then  $\omega_{k,i,u}(T(Q)) = q_{ki} - q_{k-1,u} \in \mathbb{Z}_{\geq 0}$  and  $\omega_{k,i,u}(T(Q + \delta^{ki})) = (q_{ki} + 1) - q_{k-1,u} \notin \mathbb{Z}_{\geq 0}$ , which is impossible.
- (ii) Suppose  $(p, s, u) = (k+1, s, i)$ . Then

$$\omega_{k+1,s,i}(T(Q)) = q_{k+1,s} - q_{ki} \in \mathbb{Z}_{\geq 0}$$

and

$$\omega_{k+1,s,i}(T(Q + \delta^{ki})) = q_{k+1,s} - (q_{ki} + 1) \notin \mathbb{Z}_{\geq 0}.$$

Hence  $q_{k+1,s} - q_{ki} = 0$  and then the coefficient of  $T(Q + \delta^{ki})$  in the decomposition of

$$E_{k,k+1}(T(Q)) \text{ is } - \frac{\prod_{j=1}^{k+1} (q_{ki} - q_{k+1,j})}{\prod_{j \neq i}^k (q_{ki} - q_{kj})} = 0.$$

Therefore, the tableaux that appear with nonzero coefficients in  $E_{k,k+1}(T(Q))$  are elements of  $\mathcal{N}(T(R))$ . Hence,  $E_{k,k+1}(T(Q)) \in W(T(R))$ . The proof that  $E_{k+1,k}(T(Q)) \in W(T(R))$  is analogous to the one of  $E_{k,k+1}(T(Q)) \in W(T(R))$ . The case  $g = E_{kk}$  is trivial because  $E_{kk}$  acts as a multiplication by a scalar on  $T(Q)$  and  $T(Q) \in \mathcal{N}(T(R)) \subseteq W(T(R))$ . ■

Given any tableau  $T(R)$ , there are three modules containing  $T(R)$ :  $V(T(L))$ ,  $W(T(R))$  and  $U \cdot T(R)$ . We will show that  $W(T(R)) = U \cdot T(R)$ . For this we need the following lemmas.

**Lemma 6.3.** *Let  $T(L)$  be a generic tableau. If  $0 \neq z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$  is such that  $\Omega^+(T(L)) \subseteq \Omega^+(T(L + z))$  then, there exist  $i, j$  such that  $z_{ij} \neq 0$  and*

$$\Omega^+(T(L)) \subseteq \Omega^+(T(L + z_{ij}\delta^{ij})) \subseteq \Omega^+(T(L + z)). \quad (6.1)$$

**Proof.** We will use the following definition in the proof of the lemma.

**Definition 6.4.** Given a generic tableau  $T(R) \in \mathcal{B}(T(L))$ , a *chain in  $T(R)$  of length  $\ell$  starting in row  $d$*  is a subset of the entries of  $T(R)$ ,  $C = \{r_{d-i, s^{(d-i)}}\}_{i=0, \dots, \ell}$ , where  $1 \leq s^{(d-i)} \leq d-i$  are such that  $r_{d-i, s^{(d-i)}} - r_{d-i-1, s^{(d-i-1)}} \in \mathbb{Z}$  for any  $i = 0, \dots, \ell-1$  (i.e.  $\{(d-i, s^{(d-i)}), s^{(d-i-1)}\}_{i=0, \dots, \ell} \subseteq \Omega(T(R))$ ). The chain is called maximal if

- (i)  $(d+1, i, s^{(d)}) \notin \Omega(T(R))$  for any  $1 \leq i \leq d+1$ ,
- (ii)  $(d-\ell, s^{(d-\ell)}, j) \notin \Omega(T(R))$  for any  $1 \leq j \leq d-\ell-1$ .

For every  $T(R)$  in  $\mathcal{B}(T(L))$  we have that  $\Omega^+(T(R)) = \bigsqcup_{1 \leq c \leq n} \Omega_c^+(T(R))$ , where  $\Omega_c^+(T(R)) := \{(p, s, u) \in \Omega^+(T(R)) : p = c\}$ . In particular, (6.1) holds if and only if

$$\Omega_c^+(T(L)) \subseteq \Omega_c^+(T(L + z_{ij}\delta^{ij})) \subseteq \Omega_c^+(T(L + z)) \quad (6.2)$$

for any  $1 \leq c \leq n$ . For  $c \notin \{i, i+1\}$  we have  $\Omega_c^+(T(L)) = \Omega_c^+(T(L + z_{ij}\delta^{ij}))$ . So, in order to verify (6.2), it is enough to consider the cases  $c = i, i+1$ .

Lets consider  $k, l$  such that  $z_{kl} \neq 0$ . Set for convenience  $Q := L + z$ . There exists a maximal chain  $C$  in  $T(Q)$  of length  $\ell$ , starting in row  $d$  such that  $q_{kl} \in C$ . Suppose that  $C = \{q_{[i]}\}_{i=0, \dots, \ell}$  where  $[i] := (d-i, s^{(d-i)})$ . If  $\ell = 0$ , then  $C = \{q_{kl}\}$  and (6.1) is obvious for  $z_{ij} = z_{kl}$ .

Let  $a$  and  $b$  be the minimum and maximum of  $\{i : z_{[i]} \neq 0\}$ , respectively. We have

$$\begin{aligned} \Omega_{d-a+1}^+(T(L + z_{[a]}\delta^{[a]})) &= \Omega_{d-a+1}^+(T(L + z)), \\ \Omega_{d-b}^+(T(L + z_{[b]}\delta^{[b]})) &= \Omega_{d-b}^+(T(L + z)). \end{aligned} \quad (6.3)$$

Therefore (6.2) holds for the pairs  $c = d-a+1, z_{ij} = z_{[a]}$  and  $c = d-b, z_{ij} = z_{[b]}$ , respectively. Now, let  $a \leq m \leq b$  and consider the 4 cases depending on what the signs of  $z_{[a]}$  and  $z_{[a+1]}$  are.

- (i)  $z_{[m]} > 0$  and  $z_{[m+1]} \leq 0$ . In this case (6.2) holds for  $c = d-m$  and  $z_{ij} = z_{[m]}$ . In particular, if  $z_{[a]} > 0$  and  $z_{[a+1]} \leq 0$ , using the first equation in (6.3), we conclude that (6.1) holds for  $z_{ij} = z_{[a]}$ .
- (ii)  $z_{[m]} < 0$  and  $z_{[m-1]} \geq 0$ . In this case (6.2) holds for  $c = d-m+1$  and  $z_{ij} = z_{[m-1]}$ . In particular, if  $z_{[b]} < 0$  and  $z_{[b-1]} \geq 0$ , using the second equation in (6.3) we conclude that (6.1) holds for  $z_{ij} = z_{[b]}$ .
- (iii)  $z_{[m]} > 0$  and  $z_{[m+1]} > 0$ . In this case (6.2) holds for  $c = d-m$  and

$$z_{ij} = \begin{cases} z_{[m]} & \text{if } l_{[m]} - l_{[m+1]} \in \mathbb{Z}_{\geq 0}, \\ z_{[m+1]} & \text{if } l_{[m+1]} - l_{[m]} \in \mathbb{Z}_{> 0}. \end{cases}$$

- (iv)  $z_{[m]} < 0$  and  $z_{[m-1]} < 0$ . In this case (6.2) holds for  $c = d-m+1$  and

$$z_{ij} = \begin{cases} z_{[m]} & \text{if } l_{[m-1]} - l_{[m]} \in \mathbb{Z}_{\geq 0}, \\ z_{[m-1]} & \text{if } l_{[m]} - l_{[m-1]} \in \mathbb{Z}_{> 0}. \end{cases}$$

Now combining (i)–(iv) we reduce the proof to the following two cases:

- (a)  $z_{[a]} > 0, z_{[a+1]} > 0, \dots, z_{[b]} > 0$  and for any  $t = 1, \dots, b-a$ , (6.2) holds for  $c = d-a+t+1$  and  $z_{ij} = z_{[a+t]}$ . In particular, (6.2) holds for  $c = d-b+1$  and  $z_{ij} = z_{[b]}$ . So, by the second equation in (6.3) we have that (6.1) holds for  $z_{ij} = z_{[b]}$ .
- (b)  $z_{[b]} < 0, z_{[b-1]} < 0, \dots, z_{[a]} < 0$  and for any  $t = 1, \dots, b-a$ , (6.2) holds for  $c = d-(b-t)$  and  $z_{ij} = z_{[b-t]}$ . In particular, (6.2) holds for  $c = d-a$  and  $z_{ij} = z_{[a]}$ . So, by the first equation in (6.3) we have that (6.1) holds for  $z_{ij} = z_{[a]}$ . ■



**Definition 6.5.** Given  $T(Q)$  and  $T(R)$  in  $\mathcal{B}(T(L))$ , we write  $T(R) \preceq_{(1)} T(Q)$  if there exist  $g \in \mathfrak{gl}(n)$  such that  $T(Q)$  appears with nonzero coefficient in the decomposition of  $g \cdot T(R)$  into a linear combination of tableaux. For any  $p \geq 1$  we write  $T(R) \preceq_{(p)} T(Q)$  if there exist tableaux  $T(L^{(1)}), \dots, T(L^{(p)})$ , such that

$$T(R) = T(L^{(0)}) \preceq_{(1)} T(L^{(1)}) \preceq_{(1)} \cdots \preceq_{(1)} T(L^{(p)}) = T(Q).$$

As an immediate consequence of the definition of  $\preceq_{(p)}$  we have the following.

**Lemma 6.6.** *If  $T(Q)$ ,  $T(Q^{(0)})$ ,  $T(Q^{(1)})$  and  $T(Q^{(2)})$  are tableaux in  $\mathcal{B}(T(L))$  then:*

- (i)  $T(Q^{(0)}) \preceq_{(p)} T(Q^{(1)})$  and  $T(Q^{(1)}) \preceq_{(q)} T(Q^{(2)})$  imply  $T(Q^{(0)}) \preceq_{(p+q)} T(Q^{(2)})$ ;
- (ii)  $T(Q) \preceq_{(1)} T(Q)$ .

**Corollary 6.7.** *If  $T(R), T(Q) \in \mathcal{B}(T(L))$  are generic Gelfand–Tsetlin tableaux such that  $T(R) \preceq_{(p)} T(Q)$  for some  $p \in \mathbb{Z}_{\geq 0}$ , then  $T(Q) \in U \cdot T(R)$ .*

**Proof.** By Lemma 5.4 and the definition of the relation  $\preceq_{(1)}$ , we first verify that  $T(R) \preceq_{(1)} T(Q)$  implies  $T(Q) \in U \cdot T(R)$ . Now, using Lemma 6.6(i), if  $T(R) \preceq_{(p)} T(Q)$  for some  $p$  then  $T(Q) \in U \cdot T(R)$ .  $\blacksquare$

The next theorem provides a convenient basis for the submodule of  $V(T(L))$  generated by a fixed tableau. Recall the definition of  $\mathcal{N}(T(R))$  in Notation 6.1(iii)(d).

**Theorem 6.8.** *For any tableau  $T(R) \in \mathcal{B}(T(L))$ ,  $U \cdot T(R) = W(T(R))$ . In particular,  $\mathcal{N}(T(R))$  forms a basis of  $U \cdot T(R)$ , and the action of  $\mathfrak{gl}(n)$  on  $U \cdot T(R)$  is given by the Gelfand–Tsetlin formulas.*

**Proof.** By Proposition 6.2,  $U \cdot T(R) \subseteq W(T(R))$ . To prove that  $W(T(R)) \subseteq U \cdot T(R)$  we will show that  $T(Q) \in U \cdot T(R)$  for any  $T(Q) \in \mathcal{N}(T(R))$ . By Corollary 6.7, it is enough to prove that  $T(R) \preceq_{(p)} T(Q)$  for some positive integer  $p$ .

Suppose that  $T(Q) = T(R + z) \in \mathcal{N}(T(R))$  for some  $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ . Let  $t$  be the number of non-zero components of  $z$ . We will prove that  $T(R) \preceq_{(p)} T(Q)$  using induction on  $t$ .

Let us first consider the case  $t = 1$  (the case  $t = 0$  is trivial, since then  $T(Q) = T(R)$ ) and  $z_{ij} > 0$ . We will first prove that  $T(R + l\delta^{ij}) \preceq_{(1)} T(R + (l+1)\delta^{ij})$  for any  $0 \leq l \leq z_{ij} - 1$ . This will imply

$$T(R) \preceq_{(1)} T(R + \delta^{ij}) \preceq_{(1)} T(R + 2\delta^{ij}) \preceq_{(1)} \cdots \preceq_{(1)} T(R + z_{ij}\delta^{ij}) = T(Q),$$

and then  $T(R) \preceq_{(z_{ij})} T(Q)$ . To prove that  $T(R + l\delta^{ij}) \preceq_{(1)} T(R + (l+1)\delta^{ij})$  we show that the coefficient of  $T(R + (l+1)\delta^{ij})$  in the decomposition of  $E_{i,i+1}(T(R + l\delta^{ij}))$  is not zero. In fact, by the Gelfand–Tsetlin formulas, that coefficient is

$$a_l := -\frac{\prod_{k=1}^{i+1} (r_{ij} - r_{i+1,k} + l)}{\prod_{\substack{k=1 \\ k \neq j}}^i (r_{ij} - r_{ik} + l)}.$$

Assume that  $a_l = 0$ . Then  $r_{ij} - r_{i+1,k} + l = 0$  for some  $k$ , which implies  $\omega_{i+1,k,j}(T(R)) = r_{i+1,k} - r_{ij} = l \in \mathbb{Z}_{\geq 0}$ . But, since  $T(Q) \in \mathcal{N}(T(R))$ , we have

$$l - z_{ij} = r_{i+1,k} - r_{ij} - z_{ij} = \omega_{i+1,k,j}(T(Q)) \in \mathbb{Z}_{\geq 0}.$$

Therefore we have  $0 \leq l \leq z_{ij} - 1$  and  $z_{ij} \leq l$ , which is a contradiction. Hence,  $T(R) \preceq_{(z_{ij})} T(Q)$ .

Let now  $t = 1$  and  $z_{ij} < 0$ . Using the same arguments as in the case  $z_{ij} > 0$ , we prove that  $T(R) \preceq_{(-z_{ij})} T(Q)$  using  $|z_{ij}|$  applications of  $E_{i+1,i}$ . This completes the proof for  $t = 1$ .

Assume now that for any  $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$  with at most  $t$  nonzero components, and such that  $\Omega^+(T(R)) \subseteq \Omega^+(T(R+w))$ , we have  $T(R) \preceq_{(p)} T(R+w)$  for some  $p$ . Let us consider  $z$  with  $t+1$  nonzero components. Since  $\Omega^+(T(R)) \subseteq \Omega^+(T(R+z))$ , by Lemma 6.3, there exist  $i, j$  such that

$$\Omega^+(T(R)) \subseteq \Omega^+(T(R + z_{ij}\delta^{ij})) \subseteq \Omega^+(T(R+z)).$$

Using the induction hypothesis for the pairs of tableaux  $(T(R), T(R + z_{ij}\delta^{ij}))$  and  $(T(R + z_{ij}\delta^{ij}), T(R+z))$ , there exist  $p, q \in \mathbb{Z}_{\geq 0}$  such that

$$T(R) \preceq_{(p)} T(R + z_{ij}\delta^{ij}) \quad \text{and} \quad T(R + z_{ij}\delta^{ij}) \preceq_{(q)} T(R+z).$$

Thus, by Lemma 6.6(i),  $T(R) \preceq_{(p+q)} T(R+z)$ . ■

**Proposition 6.9.** *Let  $T(R)$  and  $T(Q)$  be in  $\mathcal{B}(T(L))$ . Then  $U \cdot T(R) = U \cdot T(Q)$  if and only if  $\Omega^+(T(Q)) = \Omega^+(T(R))$ .*

**Proof.** Using Theorem 6.8 and the definitions of  $W(T(R))$ ,  $W(T(Q))$ ,  $\Omega^+(T(R))$ , and  $\Omega^+(T(Q))$ , we can prove a stronger statement:  $U \cdot T(R) \subseteq U \cdot T(Q)$  if and only if  $\Omega^+(T(Q)) \subseteq \Omega^+(T(R))$ . ■

**Corollary 6.10.**  $U \cdot T(R) = V(T(L))$  whenever  $\Omega^+(T(R)) = \emptyset$ .

**Definition 6.11.** We will write  $T(Q) \sim_{\Omega^+} T(R)$  if  $\Omega^+(T(R)) = \Omega^+(T(Q))$ .

**Proposition 6.12.** *Every submodule of  $V(T(L))$  is finitely generated.*

**Proof.** Let  $N$  be any submodule of  $V(T(L))$  and  $\Phi$  the set of all tableaux  $T(R)$  in  $N$  such that  $\Omega^+(T(P)) \subseteq \Omega^+(T(R))$  implies  $\Omega^+(T(P)) = \Omega^+(T(R))$ . By Theorem 6.8,  $N = \sum_{T(R) \in \Phi} U \cdot T(R)$

and by Proposition 6.9, we can write  $N = \bigoplus_{T(R) \in \tilde{\Phi}} U \cdot T(R)$ , where  $\tilde{\Phi}$  is a set of distinct representatives of  $\Phi / \sim_{\Omega^+}$  (hence  $\Omega^+(T(R)) \neq \Omega^+(T(Q))$  for any  $T(R), T(Q)$  in  $\tilde{\Phi}$ ). Now, since  $\Omega(T(L))$  is a finite set, then  $\tilde{\Phi}$  is finite. ■

## 6.2 Basis for irreducible modules containing a given tableau

By Theorem 6.8, the module generated by a tableau  $T(R)$  has basis  $\mathcal{N}(T(R))$ . For the purpose of the next theorem let us introduce the following equivalence on  $\mathbb{C}^{\frac{n(n+1)}{2}}$ .

**Definition 6.13.** We write  $z \sim w$  for  $z, w \in \mathbb{C}^{\frac{n(n+1)}{2}}$  if and only if one of the two cases hold.

- (i)  $z - w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$  and  $z \sim_{\Omega^+} w$ .
- (ii)  $z \in Gw$ .

Now we are ready to formulate and prove the main theorem in the paper.

**Theorem 6.14.** *The irreducible module containing  $T(R)$  has a basis of tableaux*

$$\mathcal{I}(T(R)) = \{T(Q) \in \mathcal{B}(T(R)) : \Omega^+(T(Q)) = \Omega^+(T(R))\}.$$

*The action of  $\mathfrak{gl}(n)$  on this irreducible module is given by the Gelfand–Tsetlin formulas (4.1). Therefore the set of irreducible generic Gelfand–Tsetlin modules is in one-to-one correspondence with  $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}} / \sim$ , where  $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}}$  stands for the set of generic vectors in  $\mathbb{C}^{\frac{n(n+1)}{2}}$ .*

**Proof.** For each tableau  $T(R)$ , we have an explicit construction of the module containing  $T(R)$  (recall Definition 5.6):

$$M(T(R)) := U \cdot T(R) / \left( \sum U \cdot T(Q) \right),$$

where the sum is taken over tableaux  $T(Q)$  such that  $T(Q) \in U \cdot T(R)$  and  $U \cdot T(Q)$  is a proper submodule of  $U \cdot T(R)$ .

The module  $M(T(R))$  is simple. Indeed, this follows from the fact that for any nonzero tableau  $T(S)$  in  $M(T(R))$  we have  $U \cdot T(S) = U \cdot T(R)$  and, hence,  $T(S)$  generates  $M(T(R))$ .

By Theorem 6.8 and Proposition 6.9, a basis for a proper submodule  $U \cdot T(Q)$  of  $U \cdot T(R)$  is  $\{T(S) : \Omega^+(T(R)) \subsetneq \Omega^+(T(Q)) \subseteq \Omega^+(T(S))\}$  so, a basis for the module  $\sum U \cdot T(Q)$  is  $\{T(S) : \Omega^+(T(R)) \subsetneq \Omega^+(T(S))\}$ . Therefore,  $\mathcal{I}(T(R))$  is a basis for  $M(T(R))$ .

To show that  $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}} / \sim$  parameterizes the set of all irreducible generic Gelfand–Tsetlin modules we use Theorem 5.5 and the fact that  $\ell, \ell' \in \text{Specm } \Lambda$  lie over the same  $\mathfrak{m}$  in  $\text{Specm } \Gamma$  if and only if  $\ell \in G\ell'$  (see Remark 3.3). ■

## 7 Number of irreducible modules in generic blocks

**Definition 7.1.** For any generic tableau  $T(L)$ , the *block associated with  $T(L)$*  is the set of all Gelfand–Tsetlin  $\mathfrak{gl}(n)$ -modules with Gelfand–Tsetlin support contained in  $\text{Supp}_{\text{GT}}(V(T(L)))$ .

Theorem 6.14 describe explicit bases of the irreducible modules in the block associated with  $V(T(L))$ . In this section we will use this description to compute the number of nonisomorphic irreducible modules in this block.

**Definition 7.2.** For any  $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$ ,  $1 < p \leq n$  and  $1 \leq u \leq p-1$ , define  $d_{pu}(T(R))$  to be the number of distinct elements in

$$\{r_{ps} : (p, s, u) \in \Omega(T(R))\}.$$

**Remark 7.3.** For any generic tableau  $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$  of height  $n$  we have:

- (i)  $d_{pu}(T(L)) = d_{pu}(T(R))$  for any  $1 < p \leq n$ ,  $1 \leq u \leq p-1$ ;
- (ii) if  $p \neq n$ , then  $d_{pu}(T(R)) \leq 1$  for any  $1 \leq u \leq p-1$ .

**Example 7.4.** Suppose  $a, b, c \in \mathbb{C}$  are such that  $\{a-b, a-c, b-c\} \cap \mathbb{Z} = \emptyset$ . If  $R = (a, a-1, b|a, b|c)$ , then

$$T(R) := \begin{array}{|c|c|c|} \hline a & a-1 & b \\ \hline a & b & \\ \hline c & & \\ \hline \end{array}$$

$$d_{31}(T(R)) = 2, d_{32}(T(R)) = 1, d_{21}(T(R)) = 0 \text{ and } d_{22}(T(R)) = 0.$$

**Remark 7.5.** For each tableau  $T(R)$  we have an one-to-one correspondence between the set  $\{0, 1, \dots, d_{pu}(T(L))\}$  and the subset  $\{0, i_1, \dots, i_{d_{pu}(T(L))}\}$  of  $\{0, 1, \dots, p\}$  defined as follows:  $i_1 = 1$  and  $i_k = \min\{x : r_{px} \notin \{r_{pi_1}, \dots, r_{pi_{k-1}}\}\}$ .

**Theorem 7.6.** *For any generic tableau  $T(L)$ , the number of irreducible modules in the block associated with  $T(L)$  is*

$$\prod_{1 \leq u \leq p-1 < n} (d_{pu}(T(L)) + 1).$$

*In particular,  $V(T(L))$  is irreducible if and only if  $d_{pu}(T(L)) = 0$  for any  $p$  and  $u$ , or equivalently, if and only if  $\Omega(T(L)) = \emptyset$ .*

**Proof.** By Theorem 6.14, the irreducible modules are in one-to-one correspondence with the subsets of  $\Omega(T(L))$  of the form  $\Omega^+(T(L + z))$ . For any  $T(R) \in \mathcal{B}(T(L))$ , we can decompose  $\Omega(T(R))$  into a disjoint union  $\Omega(T(R)) = \bigsqcup_{p,u} \Omega_{pu}(T(R))$ , where

$$\Omega_{p,u}(T(R)) = \{(p, 1, u), (p, 2, u), \dots, (p, p, u)\} \cap \Omega(T(R)).$$

Now, if  $\Omega_{p,u}^+(T(R)) := \Omega_{p,u} \cap \Omega^+(T(R))$ , one can write  $\Omega^+(T(R)) = \bigsqcup_{p,u} \Omega_{p,u}^+(T(R))$ . For  $p, u$  fixed, let us denote by  $s_{p,u}$  the number of different subsets of the form  $\Omega_{p,u}^+(T(R))$ . So, the number of different subsets of the form  $\Omega^+(T(R))$  is  $\prod_{p,u} s_{p,u}$ .

Let  $\{T(R^{(i)})\}_{i=1}^{s_{pu}}$  be a set of tableaux such that  $\{\Omega_{p,u}^+(T(R^{(i)}))\}_{i=1}^{s_{pu}}$  is the set of all distinct sets of the form  $\Omega_{p,u}^+(T(R))$ . We have a one-to-one correspondence between  $\{T(R^{(i)})\}_{i=1}^{s_{pu}}$  and the set  $\{0, i_1, \dots, i_{d_{pu}(T(L))}\}$  constructed as in Remark 7.5. More explicitly, this correspondence is defined by the map:

$$T(R^{(i)}) \rightarrow \begin{cases} \min\{j : (p, j, u) \in \Omega^+(T(R^{(i)}))\}, & \text{if } \Omega_{p,u}^+(T(R^{(i)})) \neq \emptyset, \\ 0, & \text{if } \Omega_{p,u}^+(T(R^{(i)})) = \emptyset. \end{cases}$$

Therefore,  $s_{pu} = d_{pu}(T(L)) + 1$ . ■

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